COMPLEX GENERALIZED GELFAND PAIRS

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Abstract

In this article we show that the pairs $(\mathrm{SO}(n,C), \mathrm{SO}(n-1,C))$ are generalized Gelfand pairs for $n \geqslant 2$.

1 Introduction

Let G be a unimodular Lie group, H a closed unimodular subgroup and let $X = G/H$. The group G acts on the space of distributions on X, denoted by $\mathcal{D}'(X)$. A continuous unitary representation π of G on a Hilbert space $\mathcal{H}$ is said to be realizable on X if there exists a G-equivariant continuous linear injection $j : \mathcal{H} \longrightarrow \mathcal{D}'(X)$. The pair (G, H) is called a generalized Gelfand pair if for all representations π with the above property, the commuting algebra of $\pi(G)$ in the algebra $\mathrm{End}(\mathcal{H})$ of all continuous linear operators of $\mathcal{H}$ into itself, is abelian. This definition generalizes the classical notion of Gelfand pair, where H is assumed compact. A direct consequence of being a Gelfand pair is the multiplicity free decomposition of $L^2(X)$ into irreducible factors.

G. van Dijk, M. T. Kosters, W. A. Kosters and M. Poel have studied several real semisimple symmetric pairs of rank one in [5], [7], [8]. They have shown that all the non-Riemannian pairs are generalized Gelfand pairs, except the pairs $(Spin(1, q+1), Spin(1, q))$ for $q \geqslant 1$, see [4]. G. van Dijk and E. P. H. Bosman also studied the p-adic analogues of some non-Riemannian pairs of rank one and they proved that they are generalized Gelfand pairs.

In this article we show that the pairs $(\mathrm{SO}(n,C), \mathrm{SO}(n-1,C))$ are generalized Gelfand pairs for $n \geqslant 2$. This result is crucial for showing that every Hilbert subspace of the space of tempered distributions $S'(C^n)$ invariant under the oscillator representation of $\mathrm{SL}(2,C) \times \mathrm{SO}(n,C)$, decomposes multiplicity free, see [1].

We could also show that the pairs $(\mathrm{SL}(n,C), \mathrm{GL}(n-1,C))$ and $(Sp(n,C), Sp(n-1,C) \times Sp(1,C))$ are generalized Gelfand pairs for $n \geqslant 3$ applying a similar method as for the real case in [7] and [8]. The difference is that we had to introduce two differential operators instead of only one. We do not include the proof in this article.

2 Definition of Generalized Gelfand Pairs

We shall give a brief summary of the theory of invariant Hilbert subspaces and generalized Gelfand pairs, for more details see [5]. Let G be a Lie group and H a closed subgroup of G. We shall assume both G and H to be unimodular. Denote by $\mathcal{D}(G)$, $\mathcal{D}(G/H)$ the space of C^∞-functions with compact support on G and G/H respectively, endowed with the usual topology. Let $\mathcal{D}'(G)$, $\mathcal{D}'(G/H)$ be the topological anti-dual of $\mathcal{D}(G)$ and $\mathcal{D}(G/H)$ respectively, provided with the strong topology.

A continuous unitary representation π of G on a Hilbert space $\mathcal{H}$ is said to be realizable on G/H is there is a continuous linear injection $j : \mathcal{H} \longrightarrow \mathcal{D}'(G/H)$ such that

$$j\pi(g) = L_g j$$

for all $g \in G$ (L_g denotes left translation by g). The space $j(\mathcal{H})$ is called an invariant Hilbert subspace of $\mathcal{D}'(G/H)$. We shall take all scalar products anti-linear in the first and linear in the second factor.

Definition 1 *The pair (G, H) is called a generalized Gelfand pair if for each continuous unitary representation π on a Hilbert space $\mathcal{H}$, which can be realized on G/H, the commutant of $\pi(G)$ in the algebra End($\mathcal{H}$) of all continuous linear operators of $\mathcal{H}$ into itself, is abelian.*

For equivalent definitions we refer to [3] and [10]. A large class of examples is given by the Riemannian semisimple symmetric pairs and by the nilpotent symmetric pairs [3], [2].

A usefull criterion for determining generalized Gelfand pairs was given by Thomas ([10], Theorem E). We shall apply it throughout this paper. Its proof is easy and straightforward (1.c.).

Denote by $\mathcal{D}'(G, H)$ the space of right H-invariant distributions on G provided with the relative topology of $\mathcal{D}'(G)$. It is well-known that $\mathcal{D}'(G, H)$ can be identified with $\mathcal{D}'(G/H)$.

Criterion 2.1 *Let $J : \mathcal{D}'(G, H) \longrightarrow \mathcal{D}'(G, H)$ be an anti-automorphism. If $J\mathcal{H} = \mathcal{H}$ (i.e. $(J|\mathcal{H})$ anti-unitary) for all G-invariant or minimal G-invariant Hilbert subspaces of $\mathcal{D}'(G, H)$, then (G, H) is a generalized Gelfand pair.*

We shall apply it in the following form.

Criterion 2.2 *Let τ be an involutive automorphism of G which leaves H stable. Define $JT = \bar{T}^\tau$ for all $T \in \mathcal{D}'(G, H)$. If $JT = T$ for all bi-H-invariant positive-definite (or extremal positive-definite) distributions on G, then (G, H) is a generalized Gelfand pair.*

Remark 1 *T^τ is defined by $< T^\tau, f > = < T, f^\tau >$ $(f \in \mathcal{D}(G))$ and $f^\tau(g) = f(\tau(g))$ $(g \in G)$. $\bar{T}$ is defined by $< \bar{T}, f > = \overline{< T, \bar{f} >}$ $(f \in \mathcal{D}(G))$.*

An important consequence of being a generalized Gelfand pair is the multiplicity-free desintegration of the left regular representation of G on $L^2(G/H)$. So one could, more or less without ambiguity, call this the Plancherel formula for G/H. If a fixed parametrization is used for the set of irreducible unitary representations realized on G/H, there is no ambiguity at all.

Let Z denote the algebra of all analytic differential operators on G which commute with left and right translations by elements of G. Any bi-H-invariant common eigendistribution of all elements of Z is called a spherical distribution. It is a well-known consequence of Schur's Lemma that any bi-H-invariant extremal positive-definite distribution on G is spherical. Spherical distributions play an important role in the harmonic analysis on G/H.

3 Morse's Lemma

Let X be a complex analytic manifold of dimension n ($n \in N$), and $f : X \longrightarrow C$ an analytic function on X. The tangent space of X at a point x^0 will be denoted by TX_{x^0}. A point $x^0 \in X$ is called a critical point of f is the induced map $f_* : TX_{x^0} \longrightarrow TC_{f(x^0)}$ is zero. If we choose a local coordinate system $(x_1, \dots, x_n)$ in a neighborhood U of x^0 this means that

$$\frac{\partial f}{\partial x_1}(x^0) = \dots = \frac{\partial f}{\partial x_n}(x^0) = 0.$$

A critical point x^0 is called non-degenerate if and only if the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x^0)\right)$$

is non-singular.

For a critical point x of f let the Hessian $H_x f$ of f at x be the quadratic form on the tangent space $T_x X$ which is defined by

$$H_x f\left(\sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}\right) = \sum_{1 \leqslant i,j \leqslant n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) u_i u_j, \quad (u_1, \dots, u_n) \in C^n,$$

in local coordinates $(x_1, \dots, x_n)$ at x.

Theorem 3.1 *(Morse's lemma) Let $f : X \longrightarrow C$ be an analytic function from a complex manifold X into C and let x^0 be a non-degenerate critical point of f. There are local coordinates $(x_1, \dots, x_n)$ at x^0 with $(0, \dots, 0)$ corresponding to x^0 such that f can be written as*

$$f(x_1, \dots, x_n) = f(x^0) + x_1^2 + \dots + x_n^2.$$

The proof of the theorem is similar to the proof of Morse's lemma in [6] p. 146.

4 The pairs $(\mathrm{SO}(n, C), \mathrm{SO}(n-1, C))$

Assume $n \geqslant 3$.

Let $G = \mathrm{SO}(n, C)$, $H = \mathrm{SO}(n-1, C))$. The space $X = G/H$ can clearly be identified with the set of all points $x = (x_1, \dots, x_n)$ in C^n satisfying $x_1^2 + \dots + x_n^2 = 1$.

We consider the following function Q on the space X which parametrizes the H-orbits on X:

$$Q(x) = x_1.$$

Q is an H-invariant complex analytic function on X with $Q(x^0) = 1$.

Define $X(z) = \{x \in X |\ Q(x) = z\}$ for $z \in C$. Now the H-orbit structure on X is as follows:

Lemma 4.1 *a) Let $z \in C$, $z \neq 1, -1$. Then $X(z)$ is a H-orbit.*

b) $X(1)$ consits of two H-orbits: $\{x^0\}$ and $\Gamma_1 = X(1) \setminus \{x^0\}$.

c) $X(-1)$ consits of two H-orbits: $\{-x^0\}$ and $\Gamma_{-1} = X(-1) \setminus \{-x^0\}$.

In order to treat the sets $X(1)$ and $X(-1)$ separately, we choose open H-invariant sets X_{-1} and X_1 such that $X(-1) \subset X_{-1}$, $X(1) \not\subset X_{-1}$, $X(1) \subset X_1$, $X(-1) \not\subset X_1$ and $X_{-1} \cup X_1 = X$. These sets clearly exist.

The critical points of Q are x^0 and $-x^0$. Both critical points are non-degenerate.

We examine Q in the neighborhood of a critical point. Firstly, near x^0 there exists a coordinate system $\{w_1, \ldots, w_{n-1}\}$ such that

$$Q(w_1, \ldots, w_{n-1}) = 1 + w_1^2 + \cdots + w_{n-1}^2,$$

x^0 corresponding to $(0, \ldots, 0)$. The Hessian $H_{x^0}Q$ at x^0 is given by

$$H_{x^0}Q(w_1, \ldots, w_{n-1}) = -w_1^2 - \cdots - w_{n-1}^2.$$

Secondly near $-x^0$ there exists a coordinate system $\{w_1, \ldots, w_{n-1}\}$ such that

$$Q(w_1, \ldots, w_{n-1}) = -1 + w_1^2 + \cdots + w_{n-1}^2,$$

$-x^0$ corresponding to $(0, \ldots, 0)$. The Hessian $H_{-x^0}Q$ at $-x^0$ is given by

$$H_{-x^0}Q(w_1, \ldots, w_{n-1}) = -w_1^2 - \cdots - w_{n-1}^2.$$

This is due to Morse's lemma.

From the properties of Q, we deduce applying [9] the existence of a linear map M, which assigns to every $f \in \mathcal{D}(X)$ a function Mf on C such that

$$\int_X F(Q(x))f(x)dx = \int_C F(z)Mf(z)dz$$

for all $F \in \mathcal{D}(C)$. Here dx is an invariant measure on X, $dz = dxdy$ $(z = x + iy)$. $Mf(z)$ gives the mean of f over the set $X(z)$. Let $\mathcal{H} = M(\mathcal{D}(X))$ and $\mathcal{H}_i = M(\mathcal{D}(X_i))$ $(i = -1, 1)$. Using the nature of the critical points of Q and the results of [9], §6 we get:

$$\begin{aligned} \mathcal{H} &= \{\varphi + \eta_0\psi_0 + \eta_1\psi_1 |\ \varphi, \psi_0, \psi_1 \in \mathcal{D}(C)\} \\ \mathcal{H}_{-1} &= \{\varphi_0 + \eta_0\psi_0 |\ \varphi_0, \psi_0 \in \mathcal{D}(Q(X_{-1}))\} \\ \mathcal{H}_1 &= \{\varphi_1 + \eta_1\psi_1 |\ \varphi_1, \psi_1 \in \mathcal{D}(Q(X_1))\}, \end{aligned}$$

where

$$\eta_0(z) = \begin{cases} |z+1|^{n-2} & \text{if } n \text{ is even} \\ |z+1|^{n-2}\mathrm{Log}|z+1| & \text{if } n \text{ is odd} \end{cases}$$

and

$$\eta_1(z) = \begin{cases} |z-1|^{n-2} & \text{if } n \text{ is even} \\ |z-1|^{n-2}\mathrm{Log}|z-1| & \text{if } n \text{ is odd} \end{cases}.$$

If we topologize $\mathcal{H}$, $\mathcal{H}_{-1}$ and $\mathcal{H}_1$ as in [9] we have for $i = -1, 1$:

a) $M : \mathcal{D}(X_i) \longrightarrow \mathcal{H}_i$ is continuous.

b) The image of the transpose map $M' : \mathcal{H}_i' \longrightarrow \mathcal{D}'(X_i)$ is the space of H-invariant distributions on X_i. M' is injective on $\mathcal{H}_i'$, because M is surjective.

Similar properties hold for $M : \mathcal{D}(X) \longrightarrow \mathcal{H}$.

So, given an H-invariant distribution on X, there exists an element $S \in \mathcal{H}'$ such that

$$< T, \varphi > = < S, M\varphi > \tag{4.1}$$

for all $\varphi \in \mathcal{D}(X)$. Fix Haar measures dg on G and dh on H in such a way that $dg = dxdh$, symbolically. For $f \in \mathcal{D}(G)$ put

$$f^\sharp(x) = \int_H f(gh)dh \qquad (x = gH).$$

Given a bi-H-invariant distribution T_0 on G, there is an unique H-invariant distribution T on X satisfying $< T_0, f > = < T, f^\sharp >$ $(f \in \mathcal{D}(G))$. This is a well-known fact.

We are now prepared to prove that (G,H) is a generalized Gelfand pair. We apply Criterion 2.2 with $JT = \bar{T}$ $(T \in \mathcal{D}'(G,H))$. We have to show that $\bar{T} = T$ for all bi-H-invariant positive-definite distributions T on G. Since $\bar{T} = \check{T}$ for such T, we shall show the following: for any bi-H-invariant distribution T on G one has $T = \check{T}$. Here $< \check{T}, f > = < T, \check{f} >$, $\check{f}(g) = f(g^{-1})$ $(g \in G,\ f \in \mathcal{D}(G))$. In view of the relation between bi-H-invariant distributions on G and H-invariant distributions on X, and because of (4.1), this amounts to the relation

$$M[(\check{f})^\sharp] = M(f^\sharp)$$

for all $f \in \mathcal{D}(G)$. For all $F \in \mathcal{D}(\mathrm{C})$ one has

$$\begin{aligned}\int_{\mathrm{C}} F(z)M[(\check{f})^\sharp](z)dz &= \int_X F(Q(x))(\check{f})^\sharp(x)dx \\ &= \int_G F(Q(g))\check{f}(g)dg \\ &= \int_G F(Q(g^{-1}))f(g)dg\end{aligned}$$

Since $Q(g) = Q(g^{-1})$ $(g \in G)$ we get the result.

So we have shown:

Theorem 4.1 *The pairs* $(\mathrm{SO}(n,\mathrm{C}), \mathrm{SO}(n-1,\mathrm{C}))$ *are generalized Gelfand pairs for* $n \geqslant 3$.

The case $n = 2$ is easily seen to provide a generalized Gelfand pair too, since $\mathrm{SO}(2,\mathrm{C})$ is an abelian group.

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